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p-Harmonic morphisms, minimal foliations, and rigidity of metrics

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Abstract

We classify p-harmonic morphisms of twisted product type from complete simply connected manifolds and polynomial p-harmonic morphisms and holomorphic p-harmonic morphisms between Euclidean spaces. We also characterize those p-harmonic functions $f : (M, g) \to \mathbb{R}$ whose level hypersurfaces produce minimal foliations of (M, g) generalizing Baird–Eells' results on harmonic morphisms. Among applications, we show that Nil space $(\mathbb{R}^3, g_{\mathrm{Nil}})$ and Sol space $(\mathbb{R}^3, g_{\mathrm{Sol}})$ admit many 1-harmonic submersions and hence many foliations by minimal surfaces. We also prove that if a complete conformally flat non-flat metric $g_U = F^{-2}\sum_{i=1}^{m} \mathrm{d}x_i^2$ on a connected open subset U of $\mathbb{R}^m$ admits one Riemannian or $m - 1$ minimal coordinate plane foliations, then (U, g_U) must be hyperbolic space $(H^m, x_m^{-2}\sum_{i=1}^{m} \mathrm{d}x_i^2)$ up to a homothety.

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1. Preliminaries

In this paper, all our objects, manifolds, vector fields, and maps are assumed to be smooth unless otherwise stated.

For $p \in (1, \infty)$, a *p-harmonic map* is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds which is a critical point of the p-energy functional:

$$E_p(\varphi) = \frac{1}{p}\int_M |\mathrm{d}\varphi|^p \mathrm{d}x.$$

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Using the first variational formula we see that a map φ is p-harmonic if and only if its p-tension field $\tau_p(\varphi) \equiv 0$, where

$$\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2}\tau_2(\varphi) + (p-2)|\mathrm{d}\varphi|^{p-3}\,\mathrm{d}\varphi(\mathrm{grad}|\mathrm{d}\varphi|)$$

with $\tau_2(\varphi)$ denoting the tension field of φ. When $|\mathrm{d}\varphi| \neq 0$, we can write

$$\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2}\{\tau_2(\varphi) + (p-2)\,\mathrm{d}\varphi(\mathrm{grad}(\ln|\mathrm{d}\varphi|))\}. \quad (1)$$

A C^2 map $\varphi : (M, g) \to (N, h)$ is a *p-harmonic morphism* if it preserves the solutions of p-Laplace equation in the sense that for any p-harmonic function $f : U \to \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a p-harmonic function.

As a generalization of Riemannian submersions, a *horizontally weakly conformal* map is a map $\varphi : (M, g) \to (N, h)$ with the property that for each $x \in M$ at which $\mathrm{d}\varphi_x \neq 0$, the restriction $\mathrm{d}\varphi_x|_{H_x} : H_x \to T_{\varphi(x)}N$ is conformal and surjective, where H_x denotes the orthogonal complement of $V_x = \ker \mathrm{d}\,\varphi_x$ in T_xM. We call H_x the horizontal and V_x the vertical space of φ at x. Thus we have $T_xM = V_x \oplus H_x$ and φ is horizontally weakly conformal implies that there is a number $\lambda(x) \in (0, \infty)$ such that $h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)) = \lambda^2(x)g(X, Y)$ for any $X, Y \in H_x$. This, in local coordinates, is equivalent to $g^{ij}(\partial\varphi^\alpha/\partial x_i)(\partial\varphi^\beta/\partial x_j) = \lambda^2 h^{\alpha\beta} \circ \varphi$. Note that at the point $x \in M$ where $\mathrm{d}\varphi_x = 0$ we can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \to \mathbb{R}$ which is called the *dilation* of a horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is called *horizontally homothetic* if the gradient of $\lambda^2(x)$ is vertical meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field X on M.

It is well known (see [9,13,20,22]) that a non-constant map is a p-harmonic morphism if and only if it is a horizontally weakly conformal p-harmonic map. Harmonic maps and harmonic morphisms are, respectively, the well-known names for 2-harmonic maps and 2-harmonic morphisms which have been studied extensively. For a detailed account and references on harmonic morphisms we recommend the recent monograph [5] and the regularly updated bibliography [15].

2. Some classifications of *p*-harmonic morphisms

In recent years, much work has been done (see [5,15]) in constructing and classifying harmonic morphisms between certain model spaces. For example, it is proved in [21] that if $\varphi : \mathbb{R}^m \to (N^n, h)$ $(n \geq 3)$ is a non-constant harmonic morphism with totally geodesic fibers, then (N, h) is isometric to $\mathbb{R}^n$ and φ is an orthogonal projection $\mathbb{R}^m \to \mathbb{R}^n$ followed by a homothety. In [14] it is showed that $\varphi : U \to \mathbb{R}^n$ $(n \geq 3)$ is a horizontally homothetic harmonic morphism with totally geodesic fibers from a connected open subset of $\mathbb{R}^m$, then φ is the restriction of an orthogonal projection $\mathbb{R}^m \to \mathbb{R}^n$ followed by a homothety. It is also proved in [16] that any non-constant holomorphic harmonic morphism $\varphi : U \to \mathbb{C}^n$ $(n \geq 2)$ from an open and connected subset U of $\mathbb{C}^m$ is the restriction of an orthogonal projection followed by a homothety. Horizontally homothetic submersions generalize the notion of Riemannian submersions.

A horizontally homothetic submersion with totally geodesic fibers and integrable horizontal distribution can be characterized as locally the projection of a warped product onto its second factor (see [5]). More generally, a horizontal conformal submersion with totally geodesic fibers and integrable horizontal distribution is characterized in [24] as locally the projection of a twisted product onto its second factor. A horizontally homothetic submersion with totally geodesic fibers and integrable distribution is a harmonic morphism called a harmonic morphism of warped product type [5]. Recently, Svensson [28] proves that a harmonic morphism of warped product type from a complete, simply connected manifold is globally the projection of a warped product onto its second factor (the universal covering space of the target manifold) followed by the covering map. In this section, we first classify all p-harmonic morphisms of twisted product type from complete and simply connected manifolds generalizing Svensson's classification [28] on harmonic morphisms of warped product type. We then give a classification of polynomial p-harmonic morphisms and holomorphic p-harmonic morphisms between Euclidean spaces generalizing Gudmundsson and Sigurdsson's classification of harmonic morphisms by classifying horizontally homothetic maps between such spaces.

Theorem 2.1. *Let $m > n \geq 1$, $1 \leq p < \infty$, and $\varphi : (M^m, g) \to (N^n, h)$ be a submersive p-harmonic morphism with totally geodesic fibers, integrable horizontal distribution, and dilation λ. When $p = n$, φ is assumed to be onto. Suppose that (M, g) is complete and simply connected. Then, (M, g) is isometric to the twisted product $F \times_{\lambda^{-2}} \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map. Furthermore, if $p \neq n$, then φ is a harmonic morphism of warped product type and it is the projection of a warped product followed by the universal covering map.*

Proof. First, we note by Theorems 3.1 and 3.4 that φ is an n-harmonic morphism being horizontally conformal submersion with totally geodesic fibers. It follows that if $p \neq n$, then it is a horizontally homothetic harmonic morphism being p-harmonic morphism for two different p values. Using Lemma 4.8 in [28] we conclude that φ is an onto submersion in any case. Let $\mathcal{F}_V$ denote the foliation of M by the fibers of φ. It is a totally geodesic foliation by assumption. On the other hand, since the horizontal distribution of φ is integrable we have another foliation $\mathcal{F}_H$ of M whose leaves are the maximal integral manifolds of horizontal distribution. By Lemma 3.2 in [14], $\mathcal{F}_H$ is totally umbilical. Since $\mathcal{F}_V$ and $\mathcal{F}_H$ are orthogonal to each other, Theorem 1 in [26] implies that (M, g) is isometric to a twisted product $F \times_{\eta^2} \tilde{N}$ with $\mathcal{F}_V$ and $\mathcal{F}_H$ corresponding to the canonical foliations of the product $F \times \tilde{N}$, where $F = \varphi^{-1}(y)$ is a fiber and $\tilde{N}$ is an integral manifold orthogonal to F. It follows that φ factors through the projection $\pi_2 : F \times_{\eta^2} \tilde{N} \to \tilde{N}$ and a map $\pi : \tilde{N} \to N$. Since M is assumed to be simply connected $\tilde{N}$ is also simply connected. Using the fact that $\pi = \varphi|\tilde{N}$ is conformal and Lemma 3.3 in do Carmo [11] we can show that the map $\pi : \tilde{N} \to N$ is a covering map and hence $\tilde{N}$ is the universal covering space of N. Since $\tilde{N}$ is the universal covering of N and $\varphi = \pi \circ \pi_2$, a simple computation shows that $\eta = \lambda^{-1}$. Therefore, we obtain the first part of the theorem. Now if $p \neq n$, then φ is a horizontally homothetic harmonic morphism. By the last assertion of Lemma 3.2 in [14], $\mathcal{F}_H$ is spherical, i.e., each leaves is an extrinsic sphere. So by Corollary 1 in [26], (M, g) is isometric to a warped product and φ becomes the projection of a warped product onto the universal covering space of N followed by the covering map. Thus, we complete the proof of the theorem. □

From the proof of Theorem 2.1 we actually have the following corollary.

Corollary 2.2. *Let $m > n \geq 1$, and $\varphi : (M^m, g) \to (N^n, h)$ be a surjective horizontally conformal submersion with totally geodesic fibers, integrable horizontal distribution, and dilation λ from a complete and simply connected manifold. Then, (M, g) is isometric to the twisted product $F \times_{\lambda^{-2}} \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map.*

Taking into account of the curvature we obtain the following corollary.

Corollary 2.3. *Let $\varphi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ be a harmonic morphism of warped product type from a complete and simply connected manifold (M^m, g) with sectional curvature $K_M \geq 0$, then (M, g) is isometric to the Riemannian product $F \times \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map which is a homothety.*

Proof. By Theorem 2.1, we know that (M, g) is isometric to the warped product $F \times_{\lambda^{-2}} \tilde{N}$ of a fiber F of φ and the universal covering space $\tilde{N}$ of N, and φ is the projection onto $\tilde{N}$ followed by the universal covering map. It follows from [32] that a complete warped product with non-negative sectional curvature must be a Riemannian product. From this we obtain the corollary. □

Notice that in Theorem 2.1 and Corollary 2.2, besides requiring (M, g) to be complete and simply connected, the usual conditions for the de Rham type of decomposition theorem, we also assume the map φ to have integrable horizontal distribution, totally geodesic fibers, and to be onto when $p = n$. These conditions seem a bit strong yet none of them can be dropped as shown by the following examples.

Example 2.4. Let $\varphi : S^3 \to S^2$ be the well-known Hopf fibration. It is an onto harmonic morphism from complete simply connected manifold with totally geodesic fibers which are great circles. We know (see e.g. [5]) that the horizontal distribution is nowhere integrable. The Theorem does not hold in this case since, topologically, S^3 cannot be diffeomorphic to $S^1 \times S^2$, the Cartesian product of a fiber and the universal covering of S^2.

Example 2.5. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with $\varphi(x, y) = \mathrm{e}^x \cos y$. It is easily checked that φ is an onto harmonic submersion (hence a submersive harmonic morphism) from a complete simply connected space. Since the horizontal distribution is one-dimensional it is integrable. It is also easy to see that the fibers are not totally geodesic in general. The Theorem does not hold in this case, too. To see this, notice that the fiber $F = \varphi^{-1}(0)$ is not connected, so, topologically, $F \times \mathbb{R}$ cannot be diffeomorphic to $\mathbb{R}^2$.

Example 2.6. Let $\varphi : \mathbb{R}^2 \times \mathbb{R}^3 \to S^3$ be the composition of the orthogonal projection $\pi : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ followed by the inverse of the stereographic projection $\sigma : \mathbb{R}^3 \to S^3$. It follows from [24] that φ is a 3-harmonic morphism with integrable horizontal distribution and totally geodesic fibers from a complete simply connected manifold. Note that φ is not onto and the theorem does not hold in this case for the similar topological argument.

We also remark that the curvature condition in Corollary 2.3 is essential, for example, consider the hyperbolic space H^m of sectional curvature -1 in the upper half-space model. We know (see [5]) that the projection of H^m onto its boundary R^{m-1} is a harmonic morphism of warped product type from a complete and simply connected space whilst H^m is not isometric to the Riemannian product of a fiber and R^{m-1}.

Now we prove the following theorem which gives a classification of horizontally homothetic maps between Euclidean spaces.

Theorem 2.7. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ $(m > n \geq 2)$ be a horizontally homothetic map. Then φ is an orthogonal projection $\mathbb{R}^m \to \mathbb{R}^n$ followed by a homothety.*

Proof. By [14] we know that for a horizontally homothetic map $\varphi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ with dilation λ we have

$$K_M(X, Y) = \lambda^2 K_N(\tilde{X}, \tilde{Y}) - \frac{3}{4}|[X, Y]^\nu|^2 - \frac{\lambda^4}{4}\left|\mathrm{grad}_\nu\left(\frac{1}{\lambda^2}\right)\right|^2,$$

where X and Y are orthonormal horizontal vector fields on M, $\tilde{X}$ and $\tilde{Y}$ are corresponding φ-related vector fields on N, and $\mathrm{grad}_\nu(1/\lambda^2)$ denotes the vertical part of the gradient of $1/\lambda^2$. Since the sectional curvature $K_M = K_N = 0$ we see from the above equation that $[X, Y]^\nu = 0$ and $\mathrm{grad}_\nu(1/\lambda^2) = 0$ i.e., the horizontal distribution is integrable and λ is constant. Therefore, φ is, up to a homothety, a Riemannian submersion with integrable horizontal distribution. Therefore, the foliation of $\mathbb{R}^m$ determined by the fibers of φ is a Riemannian foliation. Applying Theorem 1.3 in [32] we conclude that the principal curvatures of the leaves (i.e., the fibers of φ) are zero and hence the latter are totally geodesic. Now the statement that φ is an orthogonal projection followed by a homothety follows from Corollary 2.3. □

Now we are ready to give a classification of polynomial p-harmonic morphisms and holomorphic p-harmonic morphisms between Euclidean spaces.

Theorem 2.8.

(1) *Let $m > n \geq 2$, $p \in (1, \infty)$. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial p-harmonic morphism, then either $p = 2$, and φ is a harmonic morphism; or $p \neq 2$, and φ is an orthogonal projection followed by a homothety.*
(2) *For $n > 1$, $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ is a non-constant holomorphic p-harmonic morphism for $p \in (1, \infty)$ if and only if φ is an orthogonal projection followed by a homothety.*

Proof. For (1), notice that if φ is a p-harmonic morphism then it follows from [22] that it is a horizontally weakly conformal p-harmonic map. On the other hand, since φ is polynomial map, and it is proved in [1] that any horizontally weakly conformal polynomial map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ $(m > n \geq 2)$ has to be harmonic and hence a harmonic morphism. Therefore if $p \neq 2$, then φ is both a p-harmonic and a 2-harmonic morphism, which, by [22], is possible if and only if φ is a horizontally homothetic map. Now using Theorem 2.7 we obtain the statement. For Statement (2), if $p = 2$ the proof was given in [16]. If $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ is a p-harmonic morphism for $p \neq 2$, then it is a horizontally weakly conformal p-harmonic map. Noting that a holomorphic map $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ is automatically harmonic we conclude that φ is also a harmonic morphism because it is a horizontally weakly conformal harmonic map. A similar argument shows that φ is horizontally homothetic and it is an orthogonal projection followed by a homothety by Theorem 2.7. □

Remark 2.9. Note that locally, a p-harmonic morphism $\varphi : (M^m, g) \to (N^n, h)$ is a solution to the following over-determined system of partial differential equations:

$$\operatorname{div}(|\mathrm{d}\varphi|^{p-2}\,\mathrm{d}\varphi) = 0, \qquad g^{ij}\frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_j} = \lambda^2(x)h^{\alpha\beta}\bigcirc\varphi.$$

For $p = 2$, we know (cf. [2,23,25]) that many polynomial maps $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ including the maps given by multiplications of real, complex, quaternionic, and Cayley numbers solve the above equations. Theorem 2.8 says that for $p \neq 2$, the only polynomial solution is the special linear map. However, it is proved in [24] that there are many polynomial p-harmonic morphisms when the domain is given a suitable conformally flat metric.

3. *p*-Harmonic functions and minimal foliations

The most interesting link among p-harmonic morphisms, minimal foliations, and horizontally homothetic maps between Riemannian manifolds is the following theorem whose proof underwent several steps. Baird and Eells [3] obtained the theorem for the case $p = 2$. For $p \neq 2$, Statement (I) was announced in [4], and Statement (II) is basically a rearrangement of the results in [9] (see also [30]).

Theorem 3.1. *Let $m > n \geq 2$ and $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion. If*

(I) *$p = n$, then φ is p-harmonic map if and only if $\{\varphi^{-1}(y)\}_{y\in N}$ is a minimal foliation of (M, g) of codimension n.*

(II) *If $p \neq n$, then any two of the following conditions imply the other one*:
   - (a) *φ is a p-harmonic map,*
   - (b) *$\{\varphi^{-1}(y)\}_{y\in N}$ is a minimal foliation of (M, g) of codimension n,*
   - (c) *φ is horizontally homothetic.*

As noted in [9], for $p < 1$, the p-energy E_p is not a norm and $W^{1,p}$ is not a Banach space; besides, for $p = 1$, although $W^{1,1}$ becomes a Banach space, it is impossible to derive a Euler–Lagrange equation corresponding to critical points of 1-energy. So in general, it is assumed that $p > 1$ when p-harmonic maps are studied. However, in the case $f : (M, g) \to \mathbb{R}$ we do have the 1-energy functional:

$$E_1(f) = \int_M |\nabla_f| \,\mathrm{d}x \tag{2}$$

as in [10], where the authors defined the functional for a certain class of functions defined on a domain in a Euclidean space. In fact, Bombieri et al. in [10] called the functions which are critical points of the functional the functions of least gradient and they show that the level hypersurfaces of such a function are minimal. This leads to a construction of minimal graphs which are not hyperplanes in $\mathbb{R}^m$ for $m \geq 9$ and thereby solves the famous Bernstein's problem. In conformity with the language of p-energy and p-harmonic maps we adopt the following definition.

Definition 3.2. A submersion $f : (M, g) \to \mathbb{R}$ is said be 1-*harmonic* if it is a critical point of the 1-energy functional (2) defined on all functions on M which are submersions.

Lemma 3.3. *A submersion $f : (M, g) \to \mathbb{R}$ is 1-harmonic if and only if the 1-tension field $\tau_1(f) \equiv 0$, where*

$$\tau_1(f) = \frac{\Delta f - g(\nabla f, \nabla \ln |\nabla f|)}{|\nabla f|}, \tag{3}$$

with Δf denoting the Laplacian of f with the convention that on $\mathbb{R}^m$, $\Delta f = \sum_{i=1}^m \partial^2 f/\partial x_i^2$.

Proof. It is well known (see e.g. [8]) that the Euler–Lagrange equation of the functional (2) is $\mathrm{div}(\nabla f/|\nabla f|) = 0$. It is easily checked that

$$\mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right) = \frac{\{\Delta f - g(\nabla f, \nabla \ln |\nabla f|)\}}{|\nabla f|} = \tau_1(f). \tag{4}$$

Thus we obtain the Lemma. □

Now we give the following theorem which generalizes Theorem 3.1.

Theorem 3.4. *Let $f : (M^m, g) \to \mathbb{R}$ be a submersion. Then*

(I) *f is a 1-harmonic function if and only if $\{f^{-1}(t)\}_{t\in R}$ is a foliation of (M, g) by minimal hypersurfaces.*

(II) *for $p \in (1, \infty)$, any two of the following conditions imply the other one*:
  - (a) *f is a p-harmonic function,*
  - (b) *$\{f^{-1}(t)\}_{t\in R}$ is a foliation of (M, g) by minimal hypersurfaces,*
  - (c) *f is horizontally homothetic.*

To prove Theorem 3.4 we need the following two lemmas.

Lemma 3.5. *Let* $f : (M^m, g) \to \mathbb{R}$ *be a submersion and* $\eta = \nabla f/|\nabla f|$ *denote the unit normal vector field of the level hypersurfaces of f. Then the following statements are equivalent*:

(i) *f is horizontally homothetic,*
(ii) $\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = g(\nabla f, \nabla \ln|\nabla f|) = 0$,
(iii) $\mathrm{Hess}_f(\eta, \eta) = 0$,

where $\mathrm{Hess}_f(X, Y)$ *denotes the Hessian of* f.

Proof. Since the target manifold is of dimension one and f is a submersion we see that f is a horizontally conformal submersion with dilation λ given by $\lambda^2 = |\mathrm{d}f|^2 = |\nabla f|^2 = g(\nabla f, \nabla f)$. Using local coordinates $\{x^i\}$ on M we have

$$\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = (f_k\,\mathrm{d}x^k)(g^{ij}\partial_i(\ln|\nabla f|)\partial_j) = \frac{\{g^{ik}\partial_i(|\nabla f|)f_k\}}{|\nabla f|} = \frac{g(\nabla f, \nabla|\nabla f|)}{|\nabla f|},$$

where $\partial_i = \partial/\partial x_i$, $f_k = \partial f/\partial x_k$, and Einstein convention of summation is used. On the other hand, it follows from [31, p. 106] that

$$\mathrm{Hess}_f(\eta, \eta) = \frac{g(\nabla f, \nabla\lambda)}{\lambda} = \frac{g(\nabla f, \nabla|\nabla f|)}{|\nabla f|}.$$

From the above two equations we have

$$\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = \mathrm{Hess}_f(\eta, \eta) = g(\nabla f, \nabla \ln|\nabla f|). \quad (5)$$

Note that, in general, a horizontally weakly conformal map $\varphi : (M^m, g) \to (N^n, h)$ with dilation λ given by $\lambda^2 = g^{ij}(\partial\varphi^\alpha/\partial x_i)(\partial\varphi^\beta/\partial x_j)h_{\alpha\beta} = |\mathrm{d}\varphi|^2/n$ is horizontally homothetic if $X(\lambda^2) = 0$ for any horizontal vector field X on M. One can easily check that this is equivalent to $\mathrm{d}\varphi(\mathrm{grad}(\ln|\mathrm{d}\varphi|)) = 0$. This, together with Eq. (5), proves the lemma. □

Corollary 3.6.

(I) *For* $p \in [1, \infty)$ *and a submersion* $f : (M^m, g) \to \mathbb{R}$, *the p-tension field of f is given by*

$$\tau_p(f) = |\nabla f|^{p-2}\{\Delta f + (p-2)\,\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|))\}. \quad (6)$$

(II) *For* $p, q \in [1, \infty)$, *a p-harmonic submersion* $f : (M^m, g) \to \mathbb{R}$ *is also a q-harmonic submersion for* $p \neq q$ *if and only if f is horizontally homothetic in which case it is* p_1*-harmonic for any* $p_1 \geq 1$.

Proof. By (ii) of Lemma 3.5, $\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = g(\nabla f, \nabla \ln|\nabla f|)$. Combining (1) and (3) we obtain the unified form of the formula (6) for the p-tension field of a submersion including the $p = 1$ case, which gives Statement (I). To prove Statement (II), we know from (6) that a p-harmonic submersion f is also a q-harmonic submersion if and only if

$$\Delta f + (p-2)\,\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = 0, \qquad \Delta f + (q-2)\,\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = 0.$$

It follows that $\mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = 0$ since $p \neq q$. By Lemma 3.5, f is horizontally homothetic. Conversely, if f is a horizontally homothetic p-harmonic submersion, then, by (6), $\Delta f = 0$, i.e., f is also a harmonic submersion. Using (6) again we see that f is a p_1-harmonic submersion for any $p_1 \in [1, \infty)$ hence in particular it is also a q-harmonic submersion. □

Lemma 3.7. *Let $f : (M^m, g) \to \mathbb{R}$ be a submersion and $\eta = \nabla f/|\nabla f|$ be the unit normal vector field of the level hypersurfaces of f. Let $H(\eta)$ denote the mean curvature of the level hapersurfaces. Then we have*

$$(m-1)H(\eta) = -\tau_1(f) = -\frac{\{\Delta f - \mathrm{d}f(\mathrm{grad}(\ln|\nabla f|))\}}{|\nabla f|}. \tag{7}$$

Proof. By our convention on Laplace operator Δ, Eq. (8.7) in [31] reads

$$(m-1)H(\eta) = \frac{-\Delta f + \mathrm{Hess}_f(\eta, \eta)}{|\nabla f|}.$$

Using Eqs. (3) and (5) we obtain the lemma. □

Now we proceed to prove Theorem 3.4.

Proof of Theorem 3.4. Statement (I) follows immediately from Eq. (7). To prove Statement (II), we first note that the statement is true for $p = 2$ by Eq. (7). For $p \in (1, \infty) \setminus 2$, we proceed as follows:

- (a) + (b) $\Rightarrow$ (c): Suppose that f is a p-harmonic function with $p \in (1, \infty) \setminus 2$ and that $\{f^{-1}(t)\}_{t\in R}$ is a foliation of (M, g) by minimal hypersurfaces. Then, it follows from Statement (I) that f is a 1-harmonic submersion. Since $p \neq 1$ by assumption, we apply (II) of Corollary 3.6 to conclude that f is horizontally homothetic.
- (a) + (c) $\Rightarrow$ (b): Suppose that f is p-harmonic for $p \neq 1$ and that f is horizontally homothetic. It follows from (II) of Corollary 3.6 that f is also a 1-harmonic submersion. Applying Statement (I) we obtain (b).
- (b) + (c) $\Rightarrow$ (a): It follows from (b) and Statement (I) that f is a 1-harmonic submersion; this, together with (c) and (II) of Corollary 3.6, shows that f is also a p-harmonic function for any p. This yields (a), and completes the proof of Theorem 3.4.

From Theorem 3.4 we easily obtain the following corollary which generalizes Corollary 8.16 (ii) in [31]. □

Corollary 3.8. *Let $f : (M^m, g) \to \mathbb{R}$ be a p-harmonic submersion. Then, $\{f^{-1}(t)\}_{t\in R}$ is a minimal foliation of (M, g) by level hypersurfaces if and only if* (i) $p = 1$, *or* (ii) $p \neq 1$, *and f is horizontally homothetic.*

Corollary 3.9. *If $f : (M^m, g) \to \mathbb{R}$ is a 1-harmonic submersion from an orientable Riemannian manifold, then $\{f^{-1}(t)\}_{t\in R}$ is a foliation of (M, g) with each leaf a homologically area-minimizing hypersurface.*

Proof. This is a direct consequence of Theorem 3.4 and Proposition 5.2 in [33] where it was proved that given a submersion $f : (M^m, g) \to \mathbb{R}$ from an orientable Riemannian manifold with each level hypersurface minimal, then each hypersurface is actually homologically area-minimizing. □

Corollary 3.10. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a function of the form $F(x_1, \dots, x_m) = x_m - f(x_1, \dots, x_{m-1})$, where $f : \mathbb{R}^{m-1} \to \mathbb{R}$ with $m \leq 8$. Then F is 1-harmonic if and only if it is an affine function.*

Proof. F is clearly a submersion. If it is an affine function, then it is easy to see that it is a horizontally homothetic harmonic function and hence a 1-harmonic submersion by Theorem 3.4. Conversely, if F is a 1-harmonic function, then Theorem 3.4 implies that the level hypersurface $F^{-1}(c)$ is a minimal hypersurface in R^m. Notice that the level hypersurface $F^{-1}(c)$ is nothing but the graph of the function $h : \mathbb{R}^{m-1} \to \mathbb{R}, h(x) = f(x) + c$ in $\mathbb{R}^m$ for $m \leq 8$. Now Bernstein's Theorem (see e.g. [10]) implies that $f(x)$ is an affine function, and hence $F(x_1, \dots, x_m) = x_m - f(x_1, \dots, x_{m-1})$ must be an affine function. □

In [18], the author uses the curvature argument to show that the plane $x = 0$ is a minimal but not totally geodesic surface in Nil space $(\mathbb{R}^3, g_{\rm Nil})$, one of the eight three-dimensional geometries; he then further proves that there is an isometry of Nil space carrying the plane $x = 0$ to any other parallel plane $x = c$, and hence concludes that the x-coordinate plane foliation of Nil space is a minimal foliation which is not totally geodesic. Therefore the Bernstein's theorem does not hold in Nil space. The following Theorem shows that there are abundant examples of 1-harmonic submersions and hence minimal foliations on Nil space.

Theorem 3.11. *Let $(\mathbb{R}^3, g_{\rm Nil})$ denote Nil space, where the metric with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ can be written as $g_{\rm Nil} = {\rm d}x^2 + {\rm d}y^2 + ({\rm d}z - x\,{\rm d}y)^2$. Then*

(I) *A linear function $f(x, y, z) = Ax+By+Cz$ is a 1-harmonic function if and only if $A = 0$ or $C = 0$. In other words, the foliation of $\mathbb{R}^3$ by the parallel planes $\{Ax + By + Cz = t\}_{t\in R}$ is a minimal foliation with respect to Nil metric $g_{\rm Nil}$ if and only if the normal direction of the plane is orthogonal (in Euclidean sense) either to the x-axis or the z-axis. In particular, all three coordinate plane foliations of Nil space $(\mathbb{R}^3, g_{\rm Nil})$ are minimal foliations.*

(II) *For any constants A, B, and C with $C \neq 0$, the family of quadratic polynomial functions $f(x, y, z) = Ax + By + C(z - xy/2)$ are 1-harmonic submersions, and hence the level surfaces $\{Ax+By+C(z-xy/2) = t\}_{t\in R}$ give minimal foliations of Nil space $(\mathbb{R}^3, g_{\rm Nil})$ by quadratic surfaces. In particular, Nil space $(\mathbb{R}^3, g_{\rm Nil})$ admits a minimal foliation $\{z = xy/2 + t\}_{t\in R}$ by parallel hyperbolic paraboloids.*

Proof. An easy computation gives the following components of Nil metric and the coefficients of its connection:

$$\begin{aligned}
&g_{11}=1,\quad g_{12}=g_{13}=0,\quad g_{22}=1+x^2,\quad g_{23}=-x,\quad g_{33}=1;\\
&g^{11}=1,\quad g^{12}=g^{13}=0,\quad g^{22}=1,\quad g^{23}=x,\quad g^{33}=1+x^2;\\
&\Gamma^1_{11}=\Gamma^1_{33}=0,\quad \Gamma^1_{22}=-x,\quad \Gamma^1_{23}=\tfrac{1}{2},\quad \Gamma^2_{11}=\Gamma^2_{22}=\Gamma^2_{33}=\Gamma^2_{23}=0,\\
&\Gamma^3_{11}=\Gamma^3_{22}=\Gamma^3_{23}=\Gamma^3_{33}=0,\quad \Gamma^1_{12}=0,\quad \Gamma^2_{12}=\tfrac{1}{2}x,\quad \Gamma^3_{12}=\tfrac{1}{2}(x^2-1),\\
&\Gamma^1_{13}=0,\quad \Gamma^2_{13}=-\tfrac{1}{2},\quad \Gamma^3_{13}=-\tfrac{1}{2}x. \qquad (8)
\end{aligned}$$

To prove statement (I), let $f(x,y,z)=Ax+By+Cz$, then $(f_1,f_2,f_3)=(A,B,C)$ and a straightforward computation shows that

$$\lambda^2=|\nabla f|^2=A^2+B^2+C^2(1+x^2)+2BCx$$

and

$$\begin{aligned}
\Delta f=g^{ij}f_{ij}-g^{ij}\Gamma^k_{ij}f_k&=-(g^{11}\Gamma^1_{11}+g^{22}\Gamma^1_{22}+g^{33}\Gamma^1_{33}+2g^{23}\Gamma^1_{23})f_1\\
&=-[-x+2x(\tfrac{1}{2})]f_1\equiv 0.
\end{aligned}$$

This shows that the linear function f is always harmonic in Nil space. Now applying Lemma 3.7 we see that f is 1-harmonic if and only if it is horizontally homothetic, which, by Lemma 3.5, is equivalent to

$$g(\nabla f,\nabla|\nabla f|^2)=A(2xC^2+2BC)\equiv 0.$$

From this equation we obtain Statement of (I).

For Statement (II), we first note that for $f(x,y,z)=Ax+By+C(z-xy/2)$, we have $(f_1,f_2,f_3)=(A-Cy/2,B-Cx/2,C)$, and hence f is a submersion since $C\neq 0$. A direct computation gives

$$\lambda^2=|\nabla f|^2=(A-\tfrac{1}{2}Cy)^2+(B-\tfrac{1}{2}Cx)^2+C^2(1+x^2)+2Cx(B-\tfrac{1}{2}Cx)$$

and

$$\begin{aligned}
\Delta f=g^{ij}f_{ij}-g^{ij}\Gamma^k_{ij}f_k&=g^{11}f_{11}+g^{22}f_{22}+g^{33}f_{33}+2g^{23}f_{23}\\
&\quad-(g^{11}\Gamma^1_{11}+g^{22}\Gamma^1_{22}+g^{33}\Gamma^1_{33}+2g^{23}\Gamma^1_{23})f_1=-[-x+2x(\tfrac{1}{2})]f_1\equiv 0.
\end{aligned}$$

It follows that the quadratic function $f(x,y,z)=Ax+By+C(z-xy/2)$ is a harmonic function on Nil space. It is easily checked that

$$g(\nabla f,\nabla|\nabla f|^2)=g^{ij}f_i(\lambda^2)_j\equiv 0,$$

i.e., f is also horizontally homothetic, so, by Corollary 3.6 it is also a 1-harmonic submersion. Therefore we obtain Statement of (II). In particular, when $A=B=0$, and $C=1$ we see that the foliation $\mathcal{F}=\{z=xy/2+t\}_{t\in R}$ by parallel hyperbolic paraboloids is a minimal foliation. □

I am grateful to the referee for informing me the works [6,7,27] on the study of minimal surfaces on the three-dimensional Heisenberg space which lead to the following remark.

Remark 3.12.

(i) The three-dimensional Heisenberg space is the two-step nilpotent Lie group endowed with a left-invariant metric which can be identified with $(\mathbb{R}^3, g)$, where the metric, with respect to the standard coordinates X, Y, Z in $\mathbb{R}^3$, can be written as $g = \mathrm{d}X^2 + \mathrm{d}Y^2 + [\mathrm{d}Z + (Y\,\mathrm{d}X - X\,\mathrm{d}Y)/2]^2$. We can check that $\varphi : (\mathbb{R}^3, g) \to (\mathbb{R}^3, g_{\mathrm{Nil}})$ with $\varphi(X, Y, Z) = (X, Y, Z + XY/2)$ is an isometry between the Heisenberg space and Nil space. It follows from [27] that there is no totally geodesic surface in the Heisenberg space hence all the foliations given in Theorem 3.11 are non-totally geodesic minimal foliations.

(ii) Bekkar [7] studied the minimal graph equation in the three-dimensional Heisenberg space and he showed that the hyperbolic paraboloid $Z = XY/2$ is a minimal surface in the Heisenberg space. Using the fact (see e.g. [6]) that the translations along any axes are isometries in the Heisenberg space one concludes that the family of "parallel" paraboloids produce a minimal foliation of the Heisenberg space. Note that under the isometry φ given in (i) above this foliation corresponds to the minimal foliation $\{z = xy + t | t \in \mathbb{R}\}$ in Nil space. In fact, one can check that this and the one stated in (II) of Theorem 3.11 are the only two minimal foliations in Nil space produced by parallel hyperbolic paraboloids given by $z = cxy$, where c is a constant.

4. Minimal foliations and rigidity of metrics

One fundamental problem in the study of foliations is to find Riemannian metrics on a manifold that turn a given foliation into a minimal foliation (see e.g. [17]). In this section, we study the links between the existence of minimal foliations by hypersurfaces and rigidity of metrics on Riemannian manifolds. Following [12] we call a foliation of an n-dimensional manifold with all leaves diffeomorphic to $\mathbb{R}^{n-1}$ a plane foliation.

Lemma 4.1. *Let (M^m, g) be a Riemannian manifold and $U \subset M$ be a local coordinate neighborhood with coordinates $\{x_i\}$. Let $\mathcal{C}_k$ denote the kth coordinate plane foliation $\{x_k = c\}$ of (U, g_U). Then*

(i) *$\mathcal{C}_k$ is a totally geodesic foliation if and only if $\Gamma_{ij}^k = 0$ for any $i, j \in \{1, 2, \ldots, m\} \backslash \{k\}$,*

(ii) *$\mathcal{C}_k$ is a Riemannian foliation if and only if $g^{kk} = g^{kk}(x_k)$, i.e., g^{kk} is a function of x_k alone, and*

(iii) *$\mathcal{C}_k$ is a minimal foliation if and only if*

$$g^{ij}\Gamma_{ij}^k + \frac{1}{2}\frac{g^{ik}\partial_i g^{kk}}{g^{kk}} = 0. \tag{9}$$

Proof. Let $f : U \to \mathbb{R}$, $f(x_1, \ldots, x_m) = x_k$ denote the kth coordinate function which is clearly a submersion. Note that the coordinate hypersurfaces $x_k = c$ are just the level hypersurfaces of f. It follows from [31, p. 168] that $\mathcal{C}_k$ is a totally geodesic foliation if and only if $\mathrm{Hess}_f(X, Y) = 0$ for any vector fields $X, Y \in \Gamma L$, where ΓL denotes the subbundle tangent to the leaves, i.e., the hypersurfaces. Since $f(x_1, \ldots, x_m) = x_k$ we have that $\Gamma L = \mathrm{span}\{\partial_i\}_{i\neq k}$ and that

$$\mathrm{Hess}_f(\partial_i, \partial_j) = \partial_i\partial_j f - (\nabla^{\partial_j}_{\partial_i})f = -\Gamma^l_{ij}\partial_l f = -\Gamma^k_{ij} = 0$$

for any $i, j \neq k$, which gives Statement (i). To prove Statement (ii), we note that Theorem 8.9 in [31] implies that the level hypersurfaces of a submersion $f : (M, g) \to \mathbb{R}$ form a Riemannian foliation on (M, g) if and only if $X(|\nabla f|^2) = 0$ for any vector field $X \in \Gamma L$. Form this and the fact that the coordinate function $f : U \to \mathbb{R}$, $f(x_1, \ldots, x_m) = x_k$ is a submersion with $|\nabla f|^2 = g^{kk}$ and $\Gamma L = \mathrm{span}\{\partial_i\}_{i\neq k}$ we obtain Statement (ii). Finally, by Theorem 3.4 we see that the foliation of the coordinate hypersurfaces $x_k = c$ is minimal if and only if the kth coordinate function $f(x_1, \ldots, x_m) = x_k$ is 1-harmonic. Again, an easy computation turns the 1-harmonic equation $\Delta f - \mathrm{d}f(\mathrm{grad}(\ln|\nabla f|)) = 0$ for the coordinate function $f(x_1, \ldots, x_m) = x_k$ into Eq. (9) which gives Statement (iii). □

It is well known that on $\mathbb{R}^3$ all the coordinate plane foliations are totally geodesic and hence minimal foliations with respect to the standard Euclidean metric. Theorem 3.11 shows that Nil metric also has the property that all coordinate plane foliations are minimal. The following proposition and the corollary show that Sol space, another one of the eight three-dimensional geometries, has the same property.

Proposition 4.2. *Let $(\mathbb{R}^3, g_{\mathrm{Sol}})$ denote Sol space, where the metric can be written as $g_{\mathrm{Sol}} = e^{2z}\,\mathrm{d}x^2 + e^{-2z}\,\mathrm{d}y^2 + \mathrm{d}z^2$ with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$. Then a linear function $f(x, y, z) = Ax + By + Cz$ on $(\mathbb{R}^3, g_{\mathrm{Sol}})$ is always a harmonic function, it is horizontally homothetic and hence a p-harmonic function for any $p \in [1, \infty)$ if and only if either $C = 0$ or $A = B = 0$. In particular, a plane foliation $\{Ax + By + Cz = t\}_{t\in R}$ is a minimal foliation on $(\mathbb{R}^3, g_{\mathrm{Sol}})$ if and only if either $C = 0$ or $A = B = 0$.*

Proof. A direct computation gives the following components of Sol metric and the coefficients of the connection:

$$g_{11} = e^{2z},\quad g_{22} = e^{-2z},\quad g_{33} = 1,\quad \text{all other } g_{ij} = 0;\quad g^{11} = e^{-2z},$$
$$g^{22} = e^{2z},\quad g^{33} = 1,\quad \text{all other } g^{ij} = 0;\quad \Gamma^1_{11} = \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{23} = 0;$$
$$\Gamma^2_{11} = \Gamma^2_{22} = \Gamma^2_{33} = \Gamma^2_{13} = 0;\quad \Gamma^3_{11} = -e^{2z},\quad \Gamma^3_{22} = e^{-2z},\quad \Gamma^3_{33} = 0. \tag{10}$$

Now for linear function $f(x, y, z) = Ax + By + Cz$, we have $(f_1, f_2, f_3) = (A, B, C)$:

$$\lambda^2 = |\nabla f|^2 = A^2 e^{-2z} + B^2 e^{2z} + C^2$$

and

$$\Delta f = g^{ij} f_{ij} - g^{ij}\Gamma^k_{ij} f_k = -(g^{11}\Gamma^3_{11} + g^{22}\Gamma^3_{22}) f_3 = -[e^{-2z}(-e^{2z}) + e^{2z}e^{-2z}]C \equiv 0.$$

This shows that any linear function on $(\mathbb{R}^3, g_{\mathrm{Sol}})$ is a harmonic function. By Lemma 3.5, f is horizontally homothetic if and only if

$$g(\nabla f, \nabla|\nabla f|^2) = g^{ij} f_i(\lambda^2)_j = C(-2A^2 e^{-2z} + 2B^2 e^{2z}) \equiv 0.$$

This, together with Theorem 3.4, completes the proof of the proposition. □

From Proposition 4.2 and Lemma 4.1 we obtain the following corollary.

Corollary 4.3. *Let* $(\mathbb{R}^3, g_{\mathrm{Sol}})$ *be Sol space with* $g_{\mathrm{Sol}} = \mathrm{e}^{2z}\,\mathrm{d}x^2 + \mathrm{e}^{-2z}\,\mathrm{d}y^2 + \mathrm{d}z^2$ *with respect to the standard coordinates* (x, y, z) *in* $\mathbb{R}^3$. *Then the x- and y-coordinate functions are horizontally homothetic harmonic functions (which are not Riemannian submersions) whose fibers determine totally geodesic foliations of Sol space; whilst the z-coordinate function is a harmonic Riemannian submersion whose fibers yield a minimal and Riemannian foliation which is not totally geodesic.*

Example 4.4. Let $(H^m, x_m^{-2}\sum_{i=1}^m \mathrm{d}x_i^2)$ denote the hyperbolic space of upper-half-space model with $H^m = \mathbb{R}^{m-1} \times \mathbb{R}^+$. Then

(I) All the first $m-1$-coordinate functions $f(x_1, \ldots, x_m) = x_k (k \neq m)$ are horizontally homothetic harmonic so they are 1-harmonic submersions. Therefore, all the first $m-1$-coordinate plane foliations are minimal foliations. In fact, they are all totally geodesic foliations.

(II) The x_m-coordinate plane foliation is a Riemannian foliation which is not minimal.

All the above conclusions are direct consequences of Theorem 3.4, Lemma 4.1 and the following components of metric and the coefficients of connection:

$$g_{ij} = \frac{\delta_{ij}}{x_m^2}, \qquad g^{ij} = x_m^2\delta_{ij}; \qquad \Gamma_{ii}^m = -\Gamma_{im}^i = -\Gamma_{mi}^i = \frac{1}{x_m} \quad (i \neq m),$$
$$\Gamma_{mm}^m = -\frac{1}{x_m}, \qquad \text{all other } \Gamma_{ij}^k = 0.$$

For example, for the x_m-coordinate function $f(x_1, \ldots, x_m) = x_m$, we have

$$g^{ij}\Gamma_{ij}^m + \frac{1}{2}\frac{g^{im}\partial_i g^{mm}}{g^{mm}} = (m-1)x_m \neq 0.$$

Therefore, it follows from Lemma 4.1 that the x_m-coordinate plane foliation is not minimal. However, by Lemma 4.1, it is a Riemannian foliation since $g^{mm} = x_m^2$.

Example 4.4 shows that the hyperbolic metric $x_m^{-2}\sum_{i=1}^m \mathrm{d}x_i^2$ on the connected open subset H^m of $\mathbb{R}^m$ admits $m-1$ minimal coordinate plane foliations. Our next theorem shows that, up to a homothety, the hyperbolic metric is the unique complete conformally flat non-flat metric on a connected open subset of $\mathbb{R}^m$ that has this property.

Theorem 4.5. *Let* $U \subset \mathbb{R}^m$ *be an open and connected subset,* $\{x_i\}$ *be the standard orthogonal coordinates on* $\mathbb{R}^m$. *Then,*

(I) *A conformally flat metric* $g_U = F^{-2}\sum_{i=1}^m \mathrm{d}x_i^2$ *on U admits m minimal coordinate plane foliations if and only if* $F(x)$ *is a constant, or equivalently,* g_U *is a flat metric on U.*

(II) *For* $m \geq 3$, *a complete conformally flat non-flat metric* $g_U = F^{-2}\sum_{i=1}^m \mathrm{d}x_i^2$ *on* U *admits* $m-1$ *minimal coordinate plane foliations if and only if* $(U, g_U) \equiv (H^m, x_m^{-2}\sum_{i=1}^m \mathrm{d}x_i^2)$ *up to a homothety.*

(III) *For* $m \geq 3$, *a complete conformally flat non-flat metric* $g_U = F^{-2}\sum_{i=1}^m \mathrm{d}x_i^2$ *on* U *admits one Riemannian coordinate plane foliation if and only if* $(U, g_U) \equiv (H^m, x_m^{-2}\sum_{i=1}^m \mathrm{d}x_i^2)$ *up to a homothety.*

Proof. For Statement (I), we first notice that the sufficiency is clearly true because each of the m-coordinate plane foliations of U is actually totally geodesic and hence minimal with respect to the standard Euclidean metric. Conversely, If each of the m-coordinate plane foliations is minimal, then by Lemma 4.1 we have Eq. (9) for $k = 1, 2, \ldots, m$. Using coordinate $\{x_i\}$ we can write the components of g_U and the coefficients of connection as

$$g_{ij} = F^{-2}\delta_{ij}, \qquad g^{ij} = F^2\delta_{ij}.$$

$$\Gamma_{ii}^i = -f_i = -\Gamma_{jj}^i, \qquad \Gamma_{ij}^i = \Gamma_{ji}^i = -f_j (i \neq j);$$
$$\text{all other } \Gamma_{jk}^i = 0, \quad \text{where } f = \ln F. \tag{11}$$

Substituting Eq. (11) into Eq. (9) we have $(m-1)FF_k = 0$ for any $k = 1, 2, \ldots, m$. Since $m \geq 2$ we have $F_k = 0$ for any $k = 1, 2, \ldots, m$. Therefore we conclude that F is constant, which gives Statement (I).

To prove Statement (II), we assume, without loss of generality, that the first $(m-1)$-coordinate plane foliations $\mathcal{C}_i(x_i = \text{constant for } i = 1, 2, \ldots, m-1)$ are minimal. Then, as in the above proof of Statement (I), $F_k = 0$ for any $k = 1, 2, \ldots, m-1$. It follows that $F = F(x_m)$, i.e., it depends only on x_m. This, together with (11), shows that $\Gamma_{ij}^k = 0$ for any $i, j \neq k$ $(k \neq m)$. Therefore by (i) of Lemma 4.1 we conclude that the coordinate plane foliations $\mathcal{C}_k$ $(k = 1, 2, \ldots, m-1)$ are actually totally geodesic. Since x_i are orthogonal coordinates and the conformally flat metric g_U preserves orthogonality, we conclude that $\mathcal{C}_k (k = 1, 2, \ldots, m-1)$ are mutually orthogonal families of totally geodesic hypersurfaces in (U, g_U). It was proved in [19] that an m-dimensional Riemannian manifold has negative constant sectional curvature if and only if it admits $m-1$ orthogonal families of totally geodesic hypersurfaces. Therefore, we see that $g_U = F^{-2}(x_m)\sum_{i=1}^m \mathrm{d}x_i^2$ has negative constant sectional curvature which we denote by K. On the other hand, it is well known (see e.g. [29, p. 338]) that a conformally flat metric $F^{-2}\sum_{i=1}^m \mathrm{d}x_i^2$ has constant sectional curvature K if and only if

$$f_{jk} + f_j f_k = 0, \qquad j \neq k, \tag{12}$$

$$f_{ii}^2 + f_{jj}^2 - \sum_{k \neq i,j} f_k^2 = \frac{K}{F^2}, \tag{13}$$

where $f = \ln F$. Using Eq. (13) and the fact that $F(x_1, \ldots, x_m) = F(x_m)$ and that $m \geq 3$, we have $F_m = \pm\sqrt{-K}$. Solving these equations we get $F(x_m) = \pm\sqrt{-K}x_m - b$ for some constant b. Now, by connectedness of U and completeness of g_U, we have that (U, g_U) must be either one of the upper half spaces:

$$\left(\mathbb{R}^{m-1}\times\left(\frac{b}{\sqrt{-K}},\infty\right),(\sqrt{-K}x_m-b)^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2\right),$$
$$\left(\mathbb{R}^{m-1}\times\left(-\frac{b}{\sqrt{-K}},\infty\right),(\sqrt{-K}x_m+b)^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2\right);$$

or, one of the lower half spaces

$$\left(\mathbb{R}^{m-1}\times\left(-\infty,\frac{b}{\sqrt{-K}}\right),(\sqrt{-K}x_m-b)^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2\right),$$
$$\left(\mathbb{R}^{m-1}\times\left(-\infty,\frac{-b}{\sqrt{-K}}\right),(\sqrt{-K}x_m+b)^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2\right).$$

It is easy to check that each of the above spaces is homothetic to the standard upper half-space model of the hyperbolic space $(H^m, x_m^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2)$. Thereby establishing Statement (II).

For Statement (III), we have seen in Example 4.4 that the standard hyperbolic space $(H^m, x_m^{-2}\sum_{i=1}^{m}\mathrm{d}x_i^2)$ admits one Riemannian coordinate plane foliation, i.e., the mth coordinate plane foliation. Conversely, we assume, without loss of generality, that the mth coordinate plane foliation is a Riemannian foliation. Then by (ii) of Lemma 4.1 we have that $g^{mm}=F^2$ depends only on x_m. This, together with (11), implies that $\Gamma_{ij}^k=0$ for any $i, j\neq k (k\neq m)$. Therefore by (i) of Lemma 4.1, we conclude that the first $m-1$-coordinate plane foliations $\mathcal{C}_k (k=1,2,\ldots,m-1)$ are totally geodesic, and hence minimal foliations. Applying Statement (II) we obtain Statement (III). □

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